Basic topology 2

Indicate which of the following sets are open, closed or neither, justifying the answer.

1. In $\mathbb{R}$: Let $C = [a, b)$ $H = (-\infty, b)$.
2. In $\mathbb{R}$: Let $H = (-\infty, b)$.
3. In $\mathbb{R}$: $B = (-7, 1) \cup (5, 13)$.
4. In $\mathbb{R}^2$: $B = \{(x, y) \in \mathbb{R}^2 | x > 0\}$.

Solution

1. **The set C is not open**. A set U is open if for every point x in U, there exists some radius $r > 0$ such that the open ball centered at x with radius r, denoted as $B(x, r)$, is completely contained in U. The open ball $B(x, r)$ consists of all points that are at a distance less than r from x.

   In essence, an open set does not include its boundary. If you were "standing" at any point in an open set, you could move a small amount in any direction and still remain within the set. This won't happen in a.

   The set is also not closed. To be closed it should be that $\overline{C} = C$. However, $\overline{C} = [a, b]$. Therefore, the set C is neither open nor closed.

2. **The set H is not closed**, since $H = (-\infty, b)$. **The set $H = (-\infty, b)$ is an open interval on the real line**. To demonstrate that H is open using the concept of balls, consider any point x within H. Since H includes all real numbers less than b, for any x in H, we can select a radius r such that $r < b - x$. This ensures that the open ball centered at x with radius r, denoted as $B(x, r)$, is $B(x, r) = (x - r, x + r)$, and is entirely contained within H.

   Given that $x < b$ and $r < b - x$, it follows that the upper bound of the ball, $x + r$, is less than b. Therefore, $B(x, r) \subseteq H$. As this condition is met for any point x in H, the set H is open by the definition of open sets in metric spaces.

3. To prove that $B = (-7, 1) \cup (5, 13)$ is open in $\mathbb{R}$, we must show that for every point x in B, there exists an open ball $B(x, r)$ completely contained within B.

   Consider any point x in $(-7, 1)$. Since $x > -7$ and $x < 1$, we can choose r to be $r = \min\{x + 7, 1 - x\}$. Thus, the open ball $B(x, r)$ defined as $(x - r, x + r)$ is contained within $(-7, 1)$.

   For any point x in $(5, 13)$, we select r as $r = \min\{x - 5, 13 - x\}$. This ensures $B(x, r)$ is contained within $(5, 13)$.

   Since the union of two open sets is open, and both intervals $(-7, 1)$ and $(5, 13)$ are open, their union B is also open. Therefore, $B = (-7, 1) \cup (5, 13)$ is an open set in $\mathbb{R}$. **And it is not a closed set because it does not coincide with its closure $\overline{B} = [-7, 1] \cup [5, 13]$.**

4. **The set B is the positive semiplane of x. B is not an open set,** because for all points (x_0, y_0) that are on the y-axis, we cannot form a small ball that is contained in B.

   The set B is closed, since for any of its points, including those on the y-axis, we can form a small ball that intersects with B.